Restoration of Angular Lie Algebra Symmetries from a Covariant Hamiltonian

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1 october 2001

Abstract

The sO(3) and the Lorentz algebra symmetries breaking with gauge curvatures a re studied by means of a covariant Hamiltonian. The restoration of these algebra symmetries in flat and curved spaces is performed and led to the apparition of a monopole field. Then in the context of the Lorentz algebra we consider an application to the gravitoelectromagnetism theory. In this last case a qualitative relation giving a mass spectrum for dyons is established.

1 Introduction

The concept of symmetry breaking is fundamental in science and particularly in physics. The breaking of the Lorentz algebra symmetry by gauge curvature has been studied in a recent paper [1] from a covariant Hamiltonian, defined in a tangent bundle frame. The restoration of this algebra symmetry is possible by introducing a Poincaré momentum [2] and as a consequence a Dirac monopole [3].

In the present paper we insist further on the link between the restoration of algebra symmetry and the appearance of monopoles, similarly to the case of the duality symmetry of electromagnetism which is the symmetry of the free Maxwell equations under rotations of electric and magnetic fields in U(1) gauge theories. Here we also extend this approach [1] to the case of the general relativity. The section 2 recalls the basic formalism of our approach [1, 4] and deepens it by means of a generalization of Poisson brackets taking gauge fields into account. In the section 3 we study the restoration of the sO(3) algebra symmetry in a flat and a curved space, with and without the introduction of the connexion field. In the last section we consider the restoration of the Lorentz algebra symmetry in a curved space and its possible application to the theory of gravitoelectromagnetism where a qualitative relation giving a mass spectrum for dyon is deduced.

We want to remark that the brackets we use in our formalism is connected to those introduced by Feynman in his remarkable demonstration of the Maxwell equations where he tried to develop a quantization procedure without resort to a Lagrangian or a Hamiltonian. These ideas are exposed by Dyson in an elegant publication [5]. The interpretation of the Feynman's derivation of the Maxwell equations has aroused a great interest among the physicists. In particular Tanimura [6] has generalized Feynman's derivation to a Lorentz covariant form with a scalar evolution parameter. An extension of the Tanimura's approach has been provided [7] in using the Hodge duality in order to derive the two groups of Maxwell equations with a magnetic monopole in flat and curved spaces. For his part Lee [8] has included the descriptions of relativistic and non relativistic particles in an electromagnetic field, not long after Chou [9] has established a dynamical equation for spinning particles. A rigorous mathematical interpretation of Feynman's derivation connected to the inverse problem for the Poisson dynamic has been made by Carinena *et al* [10]. Then Hojman and Shepley [11] and Hughes [12] have placed this approach in the context of the Helmholtz's inverse problem

of the calculus of variations. Finally we have also to note the recent works of Montesinos and Pérez-Lorenzana [13], Singh and Dadhich [14] and Silagadze [15] which provide new looks on the Feynman's derivation.

2 Basic formalism

Let M be a three dimensional vectorial manifold where the set of vector components is homeomorphic to $\mathbb{R}^3$. In the following of this paper we shall work with these $\mathbb{R}$-triplets (or $\mathbb{R}$-quadruplets in the case of gravitoelectromagnetism in the last part). We introduce a non commutative algebra by means of skew symmetric brackets with the following distributive properties

$$[x^i, y^j z^k] = [x^i, y^j]z^k + [x^i, z^k]y^j \text{ , and } [x^i y^j, z^k] = [y^j, z^k]x^i + [x^i, z^k]y^j, \quad (1)$$

and we require the local property

$$\left[x^i, x^j\right] = 0. \quad (2)$$

Let τ be the parameter of the group of diffeomorphisms g

$$g(\mathbb{R} \times M_3) \longrightarrow M_3 : g(\tau, x^i) = g^\tau x^i = x^i(\tau), \quad (3)$$

then the "velocity vector", $\dot{x}^i = \frac{d}{d\tau} g^\tau x^i$, associated to the particle of mass m is naturally introduced by the dynamic equation $\dot{x}^i = \frac{dx^i}{d\tau} = [x^i, H]$, where the Hamiltonian H is a priori an expandable function of x^i and $\dot{x}^i$. Under these conditions we obtain the two relations

$$\left\{ \begin{array}{l} H = \frac{1}{2} m g^{ij}(x, \dot{x}) \dot{x}_j \dot{x}_i, \\ \\ m\left[x^i, \dot{x}^j\right] = g^{ij}(x, \dot{x}), \end{array} \right. \quad (4)$$

where $g^{ij}(x, \dot{x})$ is the metric tensor of the "physical space" which is chosen for the moment as an Euclidean flat space $g^{ij}(x, \dot{x}) = \delta^{ij}$. Our construction shows clearly that this Hamiltonian is covariant in a similar way to the four dimensionnal covariant Hamiltonian introduced by Goldstein [16] in the context of electromagnetism.

From our definitions we derive the following relation for expandable functions

$$[f(x, \dot{x}), g(x, \dot{x})] = \{f(x, \dot{x}), g(x, \dot{x})\} + \left[\dot{x}^i, \dot{x}^j\right] \frac{\partial f(x, \dot{x})}{\partial \dot{x}^i} \frac{\partial h(x, \dot{x})}{\partial \dot{x}^j}, \quad (5)$$

where we have used the usual Poisson brackets formalism but acting here on functions defined on the tangent bundle space

$$\{f(x,\dot{x}), g(x,\dot{x})\} = \frac{1}{m}\left(g^{ij}\frac{\partial f(x,\dot{x})}{\partial x^i}\frac{\partial h(x,\dot{x})}{\partial \dot{x}^j} - g^{ij}\frac{\partial f(x,\dot{x})}{\partial \dot{x}^i}\frac{\partial h(x,\dot{x})}{\partial x^j}\right). \quad (6)$$

We also introduce the following notation

$$J(f,g,h) = [f,[g,h]] + [g,[h,f]] + [h,[f,g]] \quad (7)$$

in order to help us to describe the structure of the Jacobi identities.

It is easy to check that, for a particle with an electric charge q, the tensor $[\dot{x}^i, \dot{x}^j]$ is a skew symmetric tensor and will be noted $\frac{q}{m^2}\mathcal{F}^{ij}(x,\dot{x})$. It corresponds to the generalization of the only position dependent three dimensional electromagnetic tensor introduced in a preceding paper [4]. Thus in the presence of such a gauge curvature $\mathcal{F}^{ij}(x,\dot{x})$ we obtain for the Jacobi structures

$$\begin{cases} J\left(f(x), g(x), h(x)\right) = 0, \\ J\left(f(x), g(x), h(x,\dot{x})\right) = 0, \\ J\left(f(x), g(x,\dot{x}), h(x,\dot{x})\right) = \dfrac{q}{m^3}\dfrac{\partial f}{\partial x^i}\dfrac{\partial g}{\partial \dot{x}^j}\dfrac{\partial h}{\partial \dot{x}^k}\dfrac{\partial \mathcal{F}^{jk}}{\partial \dot{x}_i}, \\ J\left(f(x,\dot{x}), g(x,\dot{x}), h(x,\dot{x})\right) = \dfrac{q}{m^3}\left(\dfrac{\partial \mathcal{F}^{ij}}{\partial x_k} + \dfrac{\partial \mathcal{F}^{jk}}{\partial x_i} + \dfrac{\partial \mathcal{F}^{ki}}{\partial x_j}\right)\dfrac{\partial f}{\partial \dot{x}^i}\dfrac{\partial g}{\partial \dot{x}^j}\dfrac{\partial h}{\partial \dot{x}^k} \\ +\dfrac{q}{m^3}\left(\dfrac{\partial f}{\partial x^i}\dfrac{\partial g}{\partial \dot{x}^j}\dfrac{\partial h}{\partial \dot{x}^k} + \dfrac{\partial f}{\partial \dot{x}^k}\dfrac{\partial g}{\partial x^i}\dfrac{\partial h}{\partial \dot{x}^j} + \dfrac{\partial f}{\partial \dot{x}^j}\dfrac{\partial g}{\partial \dot{x}^k}\dfrac{\partial h}{\partial x^i}\right)\dfrac{\partial \mathcal{F}^{jk}}{\partial \dot{x}_i} \\ +\dfrac{q^2}{m^4}\left(\mathcal{F}^{kl}\dfrac{\partial \mathcal{F}^{ij}}{\partial \dot{x}^l} + \mathcal{F}^{il}\dfrac{\partial \mathcal{F}^{jk}}{\partial \dot{x}^l} + \mathcal{F}^{jl}\dfrac{\partial \mathcal{F}^{ki}}{\partial \dot{x}^l}\right)\dfrac{\partial f}{\partial \dot{x}^i}\dfrac{\partial g}{\partial \dot{x}^j}\dfrac{\partial h}{\partial \dot{x}^k}. \end{cases} \quad (8)$$

The first and the second one of these relations (8) are the usual Jacobi identities and express here the local property (2) in a Euclidean flat space $g^{ij} = \delta^{ij}$. The others express the existence of the gauge curvature and are trivially null in its absence.

It is also possible to show that the Hamiltonian is invariant under his dynamical equation.The equation of motion of a particle is then simply obtained by writing

$$m\ddot{x}^i = m\left[\dot{x}^i, H\right] = q\mathcal{F}^{ij}(x,\dot{x})\dot{x}_j. \quad (9)$$

In order to study the symmetry breaking of the sO(3) algebra we introduce the usual angular momentum $L^i = m\varepsilon^i{}_{jk}x^j\dot{x}^k$ which is a constant of the motion in absence of gauge field.

3 sO(3) algebra

One of the most important symmetry which occur in physical problems is naturally the spherical symmetry corresponding to the isotropy of the physical space which is connected to the sO(3) algebra. In the following we show that this symmetry is broken when an electromagnetic field is introduced.

3.1 sO(3) algebra without gauge curvature

No electromagnetic field implies $[\dot{x}^i, \dot{x}^j] = 0$, and the sO(3) Lie algebra defined with our brackets is then equal to the standard algebra defined in terms of our Poisson brackets

$$\begin{cases} [x^i, L^j] = \{x^i, L^j\} = \varepsilon^{ijk} x_k, \\ \\ [\dot{x}^i, L^j] = \{\dot{x}^i, L^j\} = \varepsilon^{ijk} \dot{x}_k, \\ \\ [L^i, L^j] = \{L^i, L^j\} = \varepsilon^{ijk} L_k. \end{cases} \tag{10}$$

3.2 sO(3) algebra with gauge curvature

By choosing $[\dot{x}^i, \dot{x}^j] = \frac{q}{m^2} \mathcal{F}^{ij}(x, \dot{x})$, where the field $\mathcal{F}^{ij}$ is *a priori* position and velocity dependent, we generalize the gauge theories studied recently with only position dependent fields [1, 4].

So trying to retrieve the sO(3) Lie algebra we find

$$\begin{cases} \left[x^i, L^j\right] = \left\{x^i, L^j\right\} = \varepsilon^{ijk} x_k, \\ \\ \left[\dot{x}^i, L^j\right] = \left\{\dot{x}^i, L^j\right\} + \dfrac{q}{m} \varepsilon^j{}_{kl} x^k \mathcal{F}^{il}(x, \dot{x}) \\ \qquad\qquad = \varepsilon^{ijk} \dot{x}_k + \dfrac{q}{m} \varepsilon^j{}_{kl} x^k \mathcal{F}^{il}(x, \dot{x}), \\ \\ \left[L^i, L^j\right] = \left\{L^i, L^j\right\} + q \varepsilon^i{}_{kl} \varepsilon^j{}_{ms} x^k x^m \mathcal{F}^{ls}(x, \dot{x}) \\ \qquad\qquad = \varepsilon^{ijk} L_k + q \varepsilon^i{}_{kl} \varepsilon^j{}_{ms} x^k x^m \mathcal{F}^{ls}(x, \dot{x}). \end{cases} \tag{11}$$

The sO(3) algebra is then broken by the gauge curvature and we propose to recover the standard algebra in two manners.

3.2.1 Without connection

First we use a "transformation law" which generalizes angular momentum

$$L^i \rightarrow \mathcal{L}^i = L^i + \mathcal{M}^i(x,\dot{x}), \tag{12}$$

and we naturally require the usual algebra for this new angular momentum

$$\begin{cases} [x^i, \mathcal{L}^j] = \{x^i, \mathcal{L}^j\} = \varepsilon^{ijk}x_k, \\ [\dot{x}^i, \mathcal{L}^j] = \{\dot{x}^i, \mathcal{L}^j\} = \varepsilon^{ijk}\dot{x}_k, \\ [\mathcal{L}^i, \mathcal{L}^j] = \{\mathcal{L}^i, \mathcal{L}^j\} = \varepsilon^{ijk}\mathcal{L}_k. \end{cases} \tag{13}$$

From the first relation in (13) we deduce that

$$\mathcal{M}^i(x,\dot{x}) = M^i(x), \tag{14}$$

whereas the second

$$\left[\dot{x}^i, M^j\right] = -\frac{1}{m}\frac{\partial M^j(x)}{\partial x_i} = -\frac{q}{m}\varepsilon^j{}_{kl}x^k\mathcal{F}^{il}(x,\dot{x}), \tag{15}$$

implies the important property that the gauge curvature is velocity independent

$$\mathcal{F}^{ij}(x,\dot{x}) = F^{ij}(x). \tag{16}$$

This result is a consequence of the sO(3) algebra in a flat space, it is different when we consider a curved space as we shall discuss later.

The third one gives

$$M^i = \frac{1}{2}q\varepsilon_{jkl}x^i x^k F^{jl}(x) = -q\left(\overrightarrow{r}.\overrightarrow{B}\right)x^i, \tag{17}$$

where we introduce the magnetic field $\overrightarrow{B}$ that must have the same form as the Dirac magnetic monopole field

$$\overrightarrow{B} = \frac{g}{4\pi}\frac{\overrightarrow{r}}{r^3}, \tag{18}$$

in order to be a solution of the second equation in (13).

The new quantity M introduced in (17) is the well known Poincaré momentum [2] already deduced in a preceding paper [1] and the total angular momentum is then defined by

$$\overrightarrow{\mathcal{L}} = \overrightarrow{L} - \left(\overrightarrow{r}.\overrightarrow{B} \right) \overrightarrow{r}. \tag{19}$$

We note that in the precedent derivation the following Jacobi identity is directly realized

$$J\left(x^i, \dot{x}^j, \dot{x}^k\right) = \frac{q}{m^3} \frac{\partial F^{jk}(x)}{\partial \dot{x}_i} = 0, \tag{20}$$

whereas if we impose the third Jacobi identity

$$J\left(\dot{x}^i, \dot{x}^j, \dot{x}^k\right) = \frac{q}{m^3} \left(\frac{\partial F^{ij}(x)}{\partial x_k} + \frac{\partial F^{jk}(x)}{\partial x_i} + \frac{\partial F^{ki}(x)}{\partial x_j} \right) = 0, \tag{21}$$

we retrieve the Bianchi equation that is the first group of Maxwell equations for three dimensional electromagnetic field without monopole (the Poincaré momentum is equal to zero, the monopoles are absorbed by anti-monopoles).

Remark: The Poincaré momentum can also be seen to be related to the Wess-Zumino term introduced by Witten in the case of a simple mechanical problem [17]. Indeed let a physical system, with a spatial-temporal reflection symmetry, formed by a particle of mass m constrained to move on a circle. The result is that this particle is submitted to a strength having the following form $qg\varepsilon_{ijk}x_k\dot{x}_j$, that can be understood as a Lorentz force acting on an electric charge q, due to its interaction with a magnetic monopole of magnetic charge g located at the centre of the circle. For the quantum version of this system, Witten has retrieved the Dirac quantization condition by means of topological techniques.

3.2.2 With connection

In this section we consider two "transformation laws", one corresponding to the angular momentum and the other corresponding to the velocity

$$\begin{cases} L^i \to \mathcal{L}^i = L^i + \mathcal{M}^i(x, \dot{x}), \\ \\ \dot{x}^i \to p^i = m\dot{x}^i + q\mathcal{A}^i(x, \dot{x}). \end{cases} \tag{22}$$

The second transformation law is nothing else than a Legendre transformation which is defined from the tangent bundle space to the cotangent bundle

space. The bracket between the position and the linear momentum plays now the role of a "second metric tensor" G of the space.

We get instead of (4)

$$\left[x^i, p^j\right] = G^{ij}(x, \dot{x}) = g^{ij} + \frac{q}{m}\frac{\partial \mathcal{A}^j(x, \dot{x})}{\partial \dot{x}^i}. \tag{23}$$

In this section we consider only the case where $G^{ij} = \delta^{ij}$, then $\mathcal{A}^i(x, \dot{x}) = A^i(x)$, implying that the brackets between two gauge fields is zero. The theory is abelian in the sense that $[A^i(x), A^j(x)] = 0$.

In the goal to rescue the sO(3) symmetry we want the following relations to be realized

$$\begin{cases} [p^i, p^j] = \{p^i, p^j\} = 0, \\ \\ [x^i, \mathcal{L}^j] = \{x^i, \mathcal{L}\} = \varepsilon^{ijk} x_k, \\ \\ [p^i, \mathcal{L}^j] = \{p^i, \mathcal{L}^j\} = \varepsilon^{ijk} p_k, \\ \\ [\mathcal{L}^i, \mathcal{L}^j] = \{\mathcal{L}^i, \mathcal{L}^j\} = \varepsilon^{ijk} \mathcal{L}_k. \end{cases} \tag{24}$$

In (24), the first relation gives

$$\begin{aligned} \mathcal{F}^{ij}(x, \dot{x}) &= m\left[\dot{x}^j, A^i(x)\right] - m\left[\dot{x}^i, A^j(x)\right] \\ &= \frac{\partial A^j(x)}{\partial x^i} - \frac{\partial A^i(x)}{\partial x^j} \end{aligned} \tag{25}$$

which implies $\mathcal{F}^{ij} = \mathcal{F}^{ij}(x) = F^{ij}(x)$, whereas the second relation gives

$$\mathcal{M}^i(x, \dot{x}) = M^i(x). \tag{26}$$

The third provides

$$\left[\dot{x}^i, M^j\right] = q\varepsilon^{ij}{}_k A^k + q\varepsilon^j{}_{kl} x^k \left[\dot{x}^l, A^i\right] - q\varepsilon^{jk}{}_l x_k F^{il}, \tag{27}$$

and the fourth becomes

$$\begin{aligned} \varepsilon_{ijk} M^k \;=\; & \varepsilon_{ikl} x^k \left(q\varepsilon_{jm}{}^l A^m - q\varepsilon_{jmn} x^m F^{ln} - q\left[A^l, L_j\right]\right) \\ & - q\varepsilon_{jkl} x^k \left(\varepsilon_{im}{}^l A^m - q\varepsilon_{imn} x^m F^{ln} - q\left[A^l, L_i\right]\right) \\ & + q\varepsilon_{ikl}\varepsilon_{jmn} x^k x^m F^{ln}. \end{aligned} \tag{28}$$

The new "Poincaré momentum" solution of this last equation can be expressed as

$$M^i = q\varepsilon^i{}_{jk} x^j A^k \tag{29}$$

and then we recover the usual angular momentum

$$\overrightarrow{\mathcal{L}} = \overrightarrow{L} + q\overrightarrow{r} \wedge \overrightarrow{A} = \overrightarrow{r} \wedge \overrightarrow{p}. \tag{30}$$

Note that in this approach we have no access to the Maxwell equations because the Lie algebra of the linear momentum is trivial since $J\left(p^i, p^j, p^k\right) = 0$.

3.3 Generalization to the curved space case

The covariant Hamiltonian is now

$$H = \frac{1}{2} m\, g_{ij}(x) \dot{x}^i \dot{x}^j. \tag{31}$$

From the equation of motion we obtain as in the flat space case, the following relation between bracket and metric tensor

$$m\left[x^i, \dot{x}^j\right] = g^{ij}(x). \tag{32}$$

We can generalize the notion of Poisson brackets between two functions defined on a curved tangent bundled space by

$$\{f(x,\dot{x}), h(x,\dot{x})\} = \frac{1}{m} g^{ij}(x) \left(\frac{\partial f(x,\dot{x})}{\partial x^i} \frac{\partial h(x,\dot{x})}{\partial \dot{x}^j} - \frac{\partial f(x,\dot{x})}{\partial \dot{x}^i} \frac{\partial h(x,\dot{x})}{\partial x^j} \right). \tag{33}$$

The others relations are obtained in a straightforward way

$$\begin{cases} m\left[x_i, \dot{x}^j\right] = m\, g_{ik}\left\{x^k, \dot{x}^j\right\} + m\, x^k\left\{g_{ik}, \dot{x}^j\right\} = \delta_i{}^j + \dfrac{\partial g_{ik}}{\partial x_j} x^k, \\[2ex] m\left[x^i, \dot{x}_j\right] = m\, g_{jk}\left\{x^i, \dot{x}^k\right\} = \delta^i{}_j, \\[2ex] m\left[x_i, \dot{x}_j\right] = m\, g_{ik} g_{jl}\left\{x^k, \dot{x}^l\right\} + m\, g_{jl} x^k\left\{g_{ik}, \dot{x}^l\right\} = g_{ij}(x) + \dfrac{\partial g_{ik}}{\partial x^j} x^k, \end{cases} \tag{34}$$

and the brackets between functions expandable in x^i and $\dot{x}^i$, have the form

$$[f(x,\dot{x}), h(x,\dot{x})] = \{f(x,\dot{x}), h(x,\dot{x})\} + \frac{q}{m^2}\mathcal{F}^{kl}\frac{\partial f(x,\dot{x})}{\partial \dot{x}^k}\frac{\partial h(x,\dot{x})}{\partial \dot{x}^l}, \tag{35}$$

where now the gauge curvature $\mathcal{F}^{kl}(x,\dot{x})$ is velocity dependent

$$\frac{q}{m^2}\mathcal{F}^{ij}(x,\dot{x}) = \left[g^{ik}\dot{x}_k, g^{jl}\dot{x}_l\right] = \frac{1}{m}\left(\frac{\partial g^{ki}}{\partial x_j} - \frac{\partial g^{kj}}{\partial x_i}\right)\dot{x}_k + g^{ik}g^{jl}\left[\dot{x}_k, \dot{x}_l\right]. \tag{36}$$

We define the angular momentum in a three dimensional curved space by the usual relations

$$\begin{cases} L_i = m\sqrt{g(x)}\varepsilon_{ijk}x^j\dot{x}^k \\ \quad = mE_{ijk}(x)x^j\dot{x}^k = mE_i{}^{jk}(x)x_j\dot{x}_k, \\ \\ L^i = g^{ij}(x)L_j = m\sqrt{g(x)}g^{ij}(x)\varepsilon_{jkl}x^k\dot{x}^l \\ \quad = \dfrac{m}{\sqrt{g(x)}}\varepsilon^{ijk}x_j\dot{x}_k = mE^{ijk}(x)x_j\dot{x}_k, \end{cases} \tag{37}$$

where naturally $g(x) = \det\left(g_{ij}(x)\right) = \left(\det\left(g^{ij}(x)\right)\right)^{-1}$.

If we go back to the sO(3) symmetry laws, we obtain now

$$\begin{cases} \left[x^i, L^j\right] = \left\{x^i, L^j\right\} = E^{ij}{}_k x^k, \\ \\ \left[\dot{x}^i, L^j\right] = \left\{\dot{x}^i, L^j\right\} + \dfrac{q}{m}E^j{}_{kl}x^k\mathcal{F}^{il} \\ \qquad = E^{ij}{}_k\dot{x}^k - \dfrac{1}{2}E^j{}_{kl}x^k\dot{x}^l g_{mn}\dfrac{\partial g_{mn}}{\partial x_i} + \dfrac{q}{m}E^j{}_{kl}x^k\mathcal{F}^{il}, \\ \\ \left[L^i, L^j\right] = \left\{L^i, L^j\right\} + qE^i{}_{kl}E^j{}_{mn}x^k x^m\mathcal{F}^{ln} \\ \qquad = E^{ij}{}_k L^k + \dfrac{1}{2}E^i{}_{kl}E^j{}_{mn}x^k x^m\left(\dot{x}^l g_{pq}\dfrac{\partial g^{pq}}{\partial x_n} - \dot{x}^n g_{pq}\dfrac{\partial g^{pq}}{\partial x_l}\right) \\ \qquad + qE^i_{kl}E^j{}_{mn}x^k x^m\mathcal{F}^{ln} \\ \qquad + E_{qmn}x^k x^m\dot{x}^n\left(E^j{}_{kl}\dfrac{\partial g^{iq}}{\partial x_l} - E^i{}_{kl}\dfrac{\partial g^{jq}}{\partial x_l}\right), \end{cases} \tag{38}$$

where as usual we have used the formula $\frac{\partial g(x)}{\partial x^i} = g(x)g_{jk}(x)\frac{\partial g^{jk}(x)}{\partial x^i}$.

We shall use two examples of curvatures. The first is the standard gauge electromagnetic curvature, the second comes from the electromagnetic type gravity in the parametrized post newtonian formalism (the so-called "PPN formalism") that we will see in a future section.

3.3.1 Without connection

In order to restore the sO(3) symmetry as in the "flat case" we choose the transformation law of the angular momentum

$$L^i \to \mathcal{L}^i = L^i + \mathcal{M}^i(x,\dot{x}), \tag{39}$$

and we impose as usual the relations

$$\begin{cases} [x^i, \mathcal{L}^j] = E^{ij}{}_k x^k, \\ [\dot{x}^i, \mathcal{L}^j] = E^{ij}{}_k \dot{x}^k, \\ [\mathcal{L}^i, \mathcal{L}^j] = E^{ij}{}_k \mathcal{L}^k. \end{cases} \tag{40}$$

The first equation in (40) implies as in the flat space case that the new angular momentum is velocity independent, so we note $\mathcal{M}^i(x,\dot{x}) = M^i(x)$. The second equation in (40) gives

$$\begin{aligned} \left[\dot{x}^i, M^j\right] &= -\frac{1}{2mg}\frac{\partial g}{\partial x_i}L^j - \frac{q}{m}E^j_{\ lm}x^l\mathcal{F}^{im}(x,\dot{x}) \\ &= -\frac{1}{2mg}\frac{\partial g}{\partial x_i}L^j - \frac{1}{m}\left(\frac{\partial g^{ki}}{\partial x_m} - \frac{\partial g^{km}}{\partial x_i}\right)\dot{x}_k E^j_{\ lm}x^l \\ &\quad -\frac{q}{m}E^j_{\ lm}x^l g^{ik}g^{ml}\left[\dot{x}_k, \dot{x}_l\right]. \end{aligned} \tag{41}$$

Assuming the Jacobi identity

$$J\left(x^i, \dot{x}^j, \dot{x}^k\right) = \frac{q}{m^3}\frac{\partial \mathcal{F}^{jk}(x,\dot{x})}{\partial \dot{x}_i} + \frac{1}{m^2}\left(\frac{\partial g^{ki}}{\partial x_j} - \frac{\partial g^{ji}}{\partial x_k}\right) = 0, \tag{42}$$

implies that the term $\frac{q}{m^2}F^{ij}(x) = g^{ik}g^{il}\left[\dot{x}_k, \dot{x}_l\right]$ in (36) is velocity independent whereas the whole gauge field $\mathcal{F}^{ij}$ is velocity dependent. Now, since M^i is velocity independent, the velocity dependent part of the left hand side of the equation (41) must vanish leading to the relation

$$\frac{\partial g}{\partial x_i}L^j = -2g\left(\frac{\partial g^{ki}}{\partial x_m} - \frac{\partial g^{km}}{\partial x_i}\right)\dot{x}_k E^j_{\ lm}x^l. \tag{43}$$

If we compare (43) with the definition of L^j (37) we obtain a constraint relation on the metric tensor

$$\frac{\partial g}{\partial x_i}g^{nk} = -2g\left(\frac{\partial g^{ki}}{\partial x_n} - \frac{\partial g^{kn}}{\partial x_i}\right), \forall i \neq n, \tag{44}$$

that we will not use in the following.

Equation (41) reads now

$$\left[\dot{x}^i, M^j\right] = -\frac{q}{m} E^j_{\ lm} x^l F^{im}, \tag{45}$$

leading to the same result as in the flat space case, that is, the angular momentum M is a Poincaré momentum equal to

$$M^i(x) = \frac{1}{2} q E_{jkl} x^i x^k F^{jl} = -q \left(\overrightarrow{r} \overrightarrow{B}\right) x^i. \tag{46}$$

This relation (46) still implies the presence of a Dirac magnetic monopole

$$\overrightarrow{B} = \frac{g}{4\pi} \frac{\overrightarrow{r}}{r^3}. \tag{47}$$

The result of this computation is that the sO(3) symmetry algebra, in flat as well as in curved space, is restored by introducing the same Dirac monopole.

In (41) we have used the relation

$$\frac{q}{m^2} \mathcal{F}^{ij}(x, \dot{x}) = \frac{1}{m} \left(\frac{\partial g^{ki}}{\partial x_j} - \frac{\partial g^{kj}}{\partial x_i} \right) \dot{x}_k + \frac{q}{m^2} F^{ij}(x), \tag{48}$$

which shows the link between the non abelian gauge curvature $\mathcal{F}^{ij}(x, \dot{x})$ and the usual electromagnetic type abelian gauge curvature $F^{ij}(x)$. Now, we are also able to recover the equation of motion of a particle in a curved space and in the presence of an electromagnetic field. Indeed from (4) and (48) we get the well known equation of motion

$$m\ddot{x}^i = -m\Gamma^i_{jk}(x)\dot{x}^j \dot{x}^k + qF^{ij}(x)\dot{x}_j, \tag{49}$$

where we have introduced the standard Christoffel symbols Γ^i_{jk}.

3.3.2 With connection

We use the two transformation laws and we choose the "Poincaré momentum" as in the abelian case

$$\begin{cases} \dot{x}^i \to p^i = m\dot{x}^i + q\mathcal{A}^i(x, \dot{x}), \\ \\ L^i \to \mathcal{L}^i = L^i + \mathcal{M}^i(x, \dot{x}) \\ \qquad\qquad = L^i + \left(\overrightarrow{r} \wedge q\overrightarrow{\mathcal{A}}(x, \dot{x})\right)^i = \left(\overrightarrow{r} \wedge \overrightarrow{p}\right)^i, \end{cases} \tag{50}$$

then

$$\left[x^i, p^j\right] = g^{ij}(x) + \frac{q}{m}\frac{\partial \mathcal{A}^j(x,\dot{x})}{\partial \dot{x}_i} = G^{ij}(x,\dot{x}). \tag{51}$$

We note that in this context the spatial "metric tensor" $G^{ij}(x,\dot{x})$ is also velocity dependent. If we require the relations

$$\left\{\begin{array}{l} [p^i, p^j] = \{p^i, p^j\} = 0, \\ \\ [x^i, \mathcal{L}^j] = \{x^i, \mathcal{L}^j\} = E^{ij}{}_k x^k, \\ \\ [p^i, \mathcal{L}^j] = \{p^i, \mathcal{L}^j\} = E^{ij}{}_k p^k, \\ \\ [\mathcal{L}^i, \mathcal{L}^j] = \{\mathcal{L}^i, \mathcal{L}^j\} = E^{ij}{}_k \mathcal{L}^k, \end{array}\right. \tag{52}$$

the first relation gives

$$\begin{aligned} \mathcal{F}^{ij}(x,\dot{x}) &= m\left[\dot{x}^j, \mathcal{A}^i(x,\dot{x})\right] - m\left[\dot{x}^i, \mathcal{A}^j(x,\dot{x})\right] + \left[\mathcal{A}^i(x,\dot{x}), \mathcal{A}^j(x,\dot{x})\right] \\ &= \frac{\partial \mathcal{A}^j(x,\dot{x})}{\partial x^i} - \frac{\partial \mathcal{A}^i(x,\dot{x})}{\partial x^j} \\ &\quad + \mathcal{F}^{jk}(x,\dot{x})\frac{\partial \mathcal{A}^i(x,\dot{x})}{\partial \dot{x}^k} - \mathcal{F}^{ik}(x,\dot{x})\frac{\partial \mathcal{A}^j(x,\dot{x})}{\partial \dot{x}^k} \\ &\quad + \left[\mathcal{A}^i(x,\dot{x}), \mathcal{A}^j(x,\dot{x})\right]. \end{aligned} \tag{53}$$

The third equation implies

$$\frac{1}{2} g_{kl}\frac{\partial g^{kl}}{\partial x_i}\mathcal{L}^j + E^j{}_{kl} p^l \left[\mathcal{A}^i(x,\dot{x}), x^k\right] + E^j{}_{kl} x^k p^l \left[\mathcal{A}^i(x,\dot{x}), \sqrt{g}\right] = 0, \tag{54}$$

and the fourth one gives

$$\begin{aligned} &\frac{1}{2} x^m g_{kl}\frac{\partial g^{kl}}{\partial x_n}\left(E^i{}_{mn}\mathcal{L}^j - E^j{}_{mn}\mathcal{L}^i\right) \\ &+\varepsilon^i_{kl} E^j{}_{mn} x^k x^m \left(\left[\sqrt{g}, \mathcal{A}^n(x,\dot{x})\right] p^l - \left[\sqrt{g}, \mathcal{A}^l(x,\dot{x})\right] p^n\right) \\ &+E^i{}_{kl} E^j{}_{mn}\left(\left[x^k, \mathcal{A}^n(x,\dot{x})\right] x^m p^l - \left[x^m, \mathcal{A}^l(x,\dot{x})\right] x^k p^n\right) = 0. \end{aligned} \tag{55}$$

We can easily check that the last two equations are compatible with the chosen form for the generalized angular momentum (50). These last two relations must be seen as definitions of the non abelian connexion from the metric tensor.

4 Lorentz algebra

The natural extension of the angular algebra is obviously the Lorentz algebra which is primordial in the goal to envisage the gravitation theory in the frame of general relativity. Here we study only the case without the connection field.

4.1 Non abelian case

In the curved quadri-dimensional space we obtain for the fundamental bracket relations

$$\left\{\begin{array}{l} m\,[x^\mu,\dot{x}^\nu]=g^{\mu\nu}(x),\\[2ex] m\,[x_\mu,\dot{x}_\nu]=g_{\mu\nu}(x)+x^\rho\dfrac{\partial g_{\mu\rho}(x)}{\partial x^\nu},\\[2ex] m\,[\dot{x}^\mu,\dot{x}^\nu]=\dfrac{q}{m}\mathcal{F}^{\mu\nu}(x,\dot{x}),\\[2ex] m\,[\dot{x}_\mu,\dot{x}_\nu]=\dfrac{q}{m}F_{\mu\nu}(x). \end{array}\right. \tag{56}$$

It is convenient to define the angular quadri-momentum under the form

$$L_{\mu\nu}=m\left(x_\mu\dot{x_\nu}-x_\nu\dot{x}_\mu\right), \tag{57}$$

which gives the deformed Lorentz algebra with the following structure law

$$\left\{\begin{array}{l} [x_\mu,L_{\rho\sigma}]=\{x_\mu,L_{\rho\sigma}\}=g_{\mu\sigma}x_\rho-g_{\mu\rho}x_\sigma+x_\rho x^\lambda\frac{\partial g_{\mu\lambda}(x)}{\partial x^\sigma}-x_\sigma x^\lambda\frac{\partial g_{\mu\lambda}(x)}{\partial x^\rho},\\[2ex] [\dot{x}_\mu,L_{\rho\sigma}]=\{\dot{x_\mu},L_{\rho\sigma}\}+\frac{q}{m}(F_{\mu\sigma}\dot{x_\rho}-F_{\mu\rho}\dot{x_\sigma})\\ =g_{\mu\sigma}\dot{x_\rho}-g_{\mu\rho}\dot{x_\sigma}+\dot{x}_\rho x^\lambda\frac{\partial g_{\mu\lambda}(x)}{\partial x^\sigma}-\dot{x}_\sigma x^\lambda\frac{\partial g_{\mu\lambda}(x)}{\partial x^\rho}+\frac{q}{m}(F_{\mu\sigma}\dot{x_\rho}-F_{\mu\rho}\dot{x_\sigma}),\\[2ex] [L_{\mu\nu},L_{\rho\sigma}]=\{L_{\mu\nu},L_{\rho\sigma}\}+q(x_\mu x_\rho F_{\nu\sigma}-x_\nu x_\rho F_{\mu\sigma}+x_\mu x_\sigma F_{\rho\nu}-x_\nu x_\sigma F_{\rho\mu})\\ =g_{\mu\rho}L_{\nu\sigma}-g_{\nu\rho}L_{\mu\sigma}+g_{\mu\sigma}L_{\rho\nu}-g_{\nu\sigma}L_{\rho\mu}\\ +m\left(x_\rho\dot{x}_\nu x^\lambda\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}-x_\nu\dot{x}_\rho x^\lambda\frac{\partial g_{\lambda\mu}}{\partial x^\sigma}+x_\mu\dot{x}_\rho x^\lambda\frac{\partial g_{\lambda\mu}}{\partial x^\sigma}-x_\rho\dot{x}_\mu x^\lambda\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\right.\\ \left.+x_\nu\dot{x}_\sigma x^\lambda\frac{\partial g_{\lambda\rho}}{\partial x^\mu}-x_\sigma\dot{x}_\nu x^\lambda\frac{\partial g_{\lambda\mu}}{\partial x^\rho}+x_\sigma\dot{x}_\mu x^\lambda\frac{\partial g_{\lambda\nu}}{\partial x^\rho}-x_\mu\dot{x}_\sigma x^\lambda\frac{\partial g_{\lambda\rho}}{\partial x^\nu}\right)\\ +q(x_\mu x_\rho F_{\nu\sigma}-x_\nu x_\rho F_{\mu\sigma}+x_\mu x_\sigma F_{\rho\nu}-x_\nu x_\sigma F_{\rho\mu}). \end{array}\right. \tag{58}$$

In the same spirit, as in the sO(3) algebra, we choose to restore the Lorentz symmetry with the following angular quadri-momentum transformation law

$$L_{\mu\nu} \to \mathcal{L}_{\mu\nu} = L_{\mu\nu} + \mathcal{M}_{\mu\nu}(x, \dot{x}), \tag{59}$$

and we require the usual structure

$$\left\{ \begin{array}{lll} [x_\mu, \mathcal{L}_{\rho\sigma}] & = & \{x_\mu, \mathcal{L}_{\rho\sigma}\} = g_{\mu\sigma} x_\rho - g_{\mu\rho} x_\sigma, \\ [\dot{x}_\mu, \mathcal{L}_{\rho\sigma}] & = & \{\dot{x}_\mu, \mathcal{L}_{\rho\sigma}\} = g_{\mu\sigma} \dot{x}_\rho - g_{\mu\rho} \dot{x}_\sigma, \\ [\mathcal{L}_{\mu\nu}, \mathcal{L}_{\rho\sigma}] & = & \{\mathcal{L}_{\mu\nu}, \mathcal{L}_{\rho\sigma}\} = g_{\mu\rho} \mathcal{L}_{\upsilon\sigma} - g_{\nu\rho} \mathcal{L}_{\mu\sigma} + g_{\mu\sigma} \mathcal{L}_{\rho\nu} - g_{\nu\sigma} \mathcal{L}_{\rho\mu}. \end{array} \right. \tag{60}$$

From (60) we easily deduce that the quadri-momentum $\mathcal{M}_{\mu\nu}(x, \dot{x})$ is only position dependent, $\mathcal{M}_{\mu\nu}(x, \dot{x}) = M_{\mu\nu}(x)$. We call this quadri-tensor the Poincaré tensor. Equation (60) then gives

$$[\dot{x}_\mu, M_{\rho\sigma}] = \frac{q}{m}(F_{\mu\sigma} x_\rho - F_{\mu\rho} x_\sigma). \tag{61}$$

This result (61) together with the third relation given in (60) implies

$$\begin{aligned} & g_{\mu\rho} M_{\nu\sigma} - g_{\nu\rho} M_{\mu\sigma} + g_{\mu\sigma} M_{\rho\nu} - g_{\nu\sigma} M_{\rho\mu} \\ = \; & q(F_{\nu\sigma} x_\mu x_\rho - F_{\mu\sigma} x_\nu x_\rho + F_{\rho\nu} x_\mu x_\sigma - F_{\rho\mu} x_\nu x_\sigma). \end{aligned} \tag{62}$$

In order to determine the Poincaré tensor, let us firstly consider the case $\nu = \sigma = i$, where $i = 1, 2, 3$, and sum over i. Equation (62) becomes

$$- g^i{}_\rho M_{\mu i} + g_\mu{}^i M_{\rho i} - 3M_{\rho\mu} = q(-F_{\mu i} x^i x_\rho + F_{\rho i} x_\mu x^i - F_{\rho\mu} r^2), \tag{63}$$

and with $\rho = j$ and $\mu = k$, we obtain

$$M_{ij} = q(F_{ij} x^k x_k - F_{jk} x^k x_i - F_{ki} x^k x_j). \tag{64}$$

Using the definition $M_i = \varepsilon_i{}^{jk} M_{jk}$, the same magnetic angular momentum as for the $sO(3)$ case is deduced as expected for the spatial degrees of freedom ($i = 1, 2, 3$)

$$\overrightarrow{M} = -q(\overrightarrow{r}.\overrightarrow{B})\overrightarrow{r}. \tag{65}$$

Equation (61) becomes then

$$\left\{ \begin{array}{l} [\dot{x}_k, M_{ij}] = \frac{q}{m}\left(F_{kj} x_i - F_{ki} x_j\right), \\ [\dot{x}_0, M_{ij}] = \frac{q}{m}\left(F_{0j} x_i - F_{0i} x_j\right), \end{array} \right. \tag{66}$$

and if we introduce the magnetic angular momentum (65) into the set of equations (66) we obtain

$$\left\{ \begin{array}{l} x_i B_j + x_j B_i = -x_j x^k \frac{\partial B_k}{\partial x^i}, \\ \\ F_{0j} x_i - F_{0i} x_j = \left(\overrightarrow{r} \wedge \overrightarrow{E} \right)_k = 0, \end{array} \right. \tag{67}$$

which has radial vector fields centered at the origin as solutions

$$\left\{ \begin{array}{l} \overrightarrow{B} = \dfrac{g}{4\pi} \dfrac{\overrightarrow{r}}{r^3}, \\ \\ \overrightarrow{E} = q' f(r) \overrightarrow{r}. \end{array} \right. \tag{68}$$

Then we are in presence of a Schwinger dyon of magnetic charge g and electric charge q', a priori different from q the electric charge of the particle.

We still obtain the standard Poincaré magnetic angular momentum [2]

$$\overrightarrow{M} = -\frac{qg}{4\pi} \frac{\overrightarrow{r}}{r}, \tag{69}$$

and naturally we can point out that this Poincaré angular momentum has a constant norm and as Dirac in the quantum theory context we can put this norm equal to nh.

Secondary we consider the "boost" part of (62) - for $\rho = 0$, and $\mu = j$, in the equation (63) we have for the temporal components of the Poincaré tensor

$$\mathcal{M}_{0j} = q(-F_{ji} x^i x_0 - F_{0i} x_j x^i + F_{0j} r^2), \tag{70}$$

using now the Poincaré magnetic momentum we find for its component

$$\mathcal{M}_{0j} = q \left[- \left(\overrightarrow{r} \wedge \overrightarrow{B} \right)_j x_0 - \left(\overrightarrow{r} \overrightarrow{E} \right) x_j + r^2 E_j \right] = 0. \tag{71}$$

If we require the following Jacobi identity

$$J\left(\dot{x}_\mu, \dot{x}_\nu, \dot{x}_\rho\right) = \frac{q}{m^3} \left(\frac{\partial F_{\mu\nu}(x)}{\partial x^\rho} + \frac{\partial F_{\nu\rho}(x)}{\partial x^\mu} + \frac{\partial F_{\rho\mu}(x)}{\partial x^\nu} \right) = 0, \tag{72}$$

we retrieve in particular that $div \overrightarrow{B} = 0$, then the Poincaré momentum is zero.

Remark: If we introduce the linear momentum p^μ , the computation is the same as in the previous part with

$$\begin{cases} \dot{x}^\mu \to p^\mu = m\dot{x}^\mu + q\mathcal{A}^\mu(x,\dot{x}), \\ \\ L^{\mu\nu} \to \mathcal{L}^{\mu\nu} = L^{\mu\nu} + \mathcal{M}^{\mu\nu}(x) \\ \qquad\qquad = x^\mu p^\nu - x^\nu p^\mu. \end{cases} \tag{73}$$

4.2 Brief application to gravitoelectromagnetism

It is well known that for interpreting the experimental tests of gravitation theories, the Parametrized Post-Newtonian formalism (PPN) is often used [18], where the limit of low velocities and small stresses is taken. In this formalism, gravity is described by a general type metric which contains dimensionless constants call PPN-parameters, that are powerful tools in theoretical astrophysics. This formalism was applied by Braginski *et al* [19] to propose laboratory experiments to test relativistic gravity and in particular to study gravitoelectromagnetism. They analyzed magnetic and electric type gravity using a truncated and rewritten version of the PPN formalism by deleting certain parameters not present in general relativity and all gravitational non linearities. Recently Mashhoon wrote a theoretical paper [20] where he considers several important quantities relative to this theory like field equations, gravitational Larmor theorem or stress-energy tensor. He introduced gravitoelectromagnetism which is based upon the formal analogy between gravitational Newton potential and electric Coulomb potential. A long time ago Holzmuller [21] and Tisserand [22] have already postulated gravitational electromagnetic components for the gravitational influence of the sun on the motion of planets. Finally Mashhoon [20] considers that a particle of inertial mass m has gravitoelectric charge $q_E = -m$ and gravitomagnetic charge $q_M = -2m$, the numerical factor 2 coming from the spin character of the gravitational field. In the final part of this work we apply these last ideas to our formalism.

Suppose that gravitation creates a gravitoelectromagnetic field, then we have the following equation

$$\mathcal{M}_{ij} = q(F_{ij}x^k x_k - F_{jk}x^k x_i - F_{ki}x^k x_j) + g({}^*F_{ij}x^k x_k - {}^*F_{jk}x^k x_i - {}^*F_{ki}x^k x_j). \tag{74}$$

Here we introduce the Hodge duality such as

$$[\dot{x}_\mu, \dot{x}_\nu] = -\frac{1}{m^2}(qF_{\mu\nu} + g^*F_{\mu\nu}), \tag{75}$$

g being the magnetic charge of the particle, which can be seen now as a Schwinger dyon.

The new angular momentum is the sum of two contributions, a gravitomagnetic and a gravitoelectric one

$$\overrightarrow{M} = -q(\overrightarrow{r}.\overrightarrow{B})\overrightarrow{r} + g(\overrightarrow{r}.\overrightarrow{E})\overrightarrow{r} = \overrightarrow{M_m} + \overrightarrow{M_e}, \tag{76}$$

where

$$\begin{cases} \overrightarrow{M_m} = -q(\overrightarrow{r}.\overrightarrow{B})\overrightarrow{r}, \\ \\ \overrightarrow{M_e} = g(\overrightarrow{r}.\overrightarrow{E})\overrightarrow{r} \end{cases} \tag{77}$$

are the gravitomagnetic and gravitoelectric angular momenta. Due to the fact that for these gravitational monopoles the source of the fields is localized at the origin, we obtain for the vector $\overrightarrow{P} = q\overrightarrow{B} - g\overrightarrow{E}$

$$\begin{aligned} div\overrightarrow{P} &= m^3 J\left(\dot{x}^i, \dot{x}^j, \dot{x}^k\right) = q\, div\overrightarrow{B} - g\, div\overrightarrow{E} \\ &= \frac{qg}{4\pi}\left[x^l, \frac{x_l}{r^3}\right] = qg\,\delta^3(\overrightarrow{r}). \end{aligned} \tag{78}$$

So as an example

$$\begin{cases} \overrightarrow{B} = \dfrac{g'}{4\pi}\dfrac{\overrightarrow{r}}{r^3}, \\ \\ \overrightarrow{E} = -\dfrac{q'}{4\pi}\dfrac{\overrightarrow{r}}{r^3}, \end{cases} \tag{79}$$

where a priori g' and q' are different from g and q. If we require now the Jacobi identity between the velocities

$$J\left(\dot{x}^\mu, \dot{x}^\nu, \dot{x}^\rho\right) = 0, \tag{80}$$

we have the generalized gravitational Maxwell equations

$$q(\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu}) + g(\partial^{\mu*}F^{\nu\rho} + \partial^{\nu*}F^{\rho\mu} + \partial^{\rho*}F^{\mu\nu}) = 0. \tag{81}$$

The projection of (81) on three dimensional space gives

$$q\, div\overrightarrow{B} - g\, div\overrightarrow{E} = div\overrightarrow{P} = 0, \tag{82}$$

where $\overrightarrow{P}$ can be taken either perpendicular to the vector $\overrightarrow{r}$ or null, we then have $\overrightarrow{M} = \overrightarrow{0}$.

If we don't require this identity, we have

$$q(\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu}) + g(\partial^{\mu *}F^{\nu\rho} + \partial^{\nu *}F^{\rho\mu} + \partial^{\rho *}F^{\mu\nu}) = qgN^{\mu\nu\rho}, \quad (83)$$

where $N^{\mu\nu\rho}$ is the components of a 3-differential form N which breaks this identity.

In both cases (Jacobi identity required or not) we obtain the following two groups of gravitational Maxwell equations [4]

$$\begin{cases} \delta F = j, \\ \\ dF = -^{*}k, \end{cases} \quad (84)$$

but in the case where the Jacobi identity is not required we have the relation $g^{*}j - q^{*}k = qgN$, and for the case where the Jacobi identity is required this relation is replaced by $g^{*}j = q^{*}k$.

Then in the frame of gravitoelectromagnetism we have for the two charges (q, g) of the dyon particle and for the two charges (q', g') of the gravitoelectromagnetic dyon similar to the one studied by Mashhoon [20]

$$\begin{cases} q &= \alpha \dfrac{q_E}{\varepsilon_0}, \\ \\ g &= \beta \mu_0 q_M, \\ \\ q' &= \alpha' \dfrac{q'_E}{\varepsilon_0}, \\ \\ g' &= \beta' \mu_0 q'_M, \end{cases} \quad (85)$$

where ε_0 and μ_0 are respectively the vacuum permittivity and permeability, and α, β, α' and β' are constants with the following dimensional relations

$$\begin{cases} [\alpha] = [\alpha'] = L^{-2}, \\ \\ [\beta] = [\beta'] = L^{-1}T^{-1}. \end{cases} \quad (86)$$

From the quantization of the relation (76) we deduce

$$\alpha\beta' q_E q'_M + \alpha'\beta q'_E q_M = \frac{2nh}{Z_0^2}, \quad (87)$$

where Z_0 is the vacuum impedance, and if we postulate as in Mashhoon's paper [20] the following relations

$$\begin{cases} q_E &= a\sqrt{G}m, \\ q_M &= b\sqrt{G}m, \\ q'_E &= a'\sqrt{G}m', \\ q'_M &= b'\sqrt{G}m', \end{cases} \tag{88}$$

where a, b, a' and b' are constants, m' is the monopole mass *a priori* different from the mass m of the particle and G is the gravitational constant. We have also $\frac{q_M}{q_E} = \frac{q'_M}{q'_E} = \frac{b}{a} = \frac{b'}{a'} = s$ which is the Mashhoon's relation between electric and magnetic charges and the spin of the gauge boson interaction. In the gravitoelectromagnetic theory we naturally have $s = 2$, then we deduce

$$m\,m' = \frac{nh}{aa'(\alpha\beta' + \alpha'\beta)GZ_0^2} = \frac{nh}{AGZ_0^2}, \tag{89}$$

where A is a new constant and n is an integer number in the Schwinger formalism (bosonic spectrum) and half integer number in the Dirac formalism (fermionic and bosonic spectrum). We have then obtain a qualitative relation which gives the mass spectrum of dyon particles and, in the particular case where $q = q'$ and $g = g'$, it simply becomes

$$m = m' = \sqrt{\frac{nh}{AGZ_0^2}}. \tag{90}$$

5 Conclusion

We have studied the breaking of sO(3) and Lorentz algebras symmetries by "abelian" or "non abelian" gauge curvature in a covariant formalism. The restoration of these algebra symmetries is made in two different ways: the first one with the introduction of a generalized angular momentum which is the Poincaré momentum and the second one with a Legendre transformation of the velocity that naturally introduces a linear momentum and a connexion. We pointed out that it is more interesting to work in the tangent bundle

approach if we want to study the equations of motion and the existence of a monopole than in the cotangent bundle approach. The application of this formalism to the gravitoelectromagnetism theory was envisaged. The principal result of this last part is the extrapolation of the Dirac quantum condition for the magnetic monopole to the case of a potential gravitoelectromagnetic monopole. Finally using Mashhoon's postulate for the relation between the gravitocharges and the mass of the particle, we obtain a qualitative condition on the mass spectrum.

Actually we look at the theory of spinning point particles in different terms than Chou [9] , then we retrieve in particular the principal results that Van Holten [23] has found in an other context.

Acknowledgments

A.B. would like to thank Patrice Pérez for helpful discussions.

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